Stat 512 — Take home exam I (due on July 10th)

1. Let Y has an exponential distribution with mean β. Prove that $W = \sqrt{Y}$ has a Weibull density. (Hint: Use the C.D.F technique; the pdf of Weibull is: $f_W(w) = \frac{k}{\lambda}\left(\frac{w}{\lambda}\right)^{k-1} e^{-(w/\lambda)^k}, w \geq 0$; identify k and λ) (20 pts)

2. Let $Y_1, Y_2 \sim N(0,1)$, Y_1, Y_2 are independent random variable. Find the distribution of $U = \frac{Y_1}{Y_2}$. (Hint: Using transformation technique) (20 pts)

3. Suppose that a unit of mineral ore contains a proportion Y_1 of metal A and a Proportion Y_2 of metal B. Experience has shown that the joint probability density function of Y_1 and Y_2 is uniform over the region $0 \leq y_1 \leq 1, 0 \leq y_2 \leq 1, 0 \leq y_1 + y_2 \leq 1$. Let $U = Y_1 + Y_2$, the proportion of either metal A or B per unit. Find

   a) the probability density function for U (use the CDF technique). (10 pts)

   b) $E(U)$ and $Var(U)$ by using the answer to part (a). (10 pts)

4. The total time from arrival to completion of service at a fast-food outlet , Y_1 and the time spent waiting in line before arriving at the service window, Y_2, have a joint density function:

$$f(y_1, y_2) = \begin{cases} e^{-y_1}, & 0, \leq y_2 \leq y_1 < \infty \\ 0, & o.w \end{cases}$$

   Another random variable of interest is $U = Y_1 - Y_2$, the time spent at the service window. Find:

   a) the probability density function for U using bivariate transformation technique. (10 pts)

   b) $E(U)$ and $Var(U)$ by using the answer to part (a). (10 pts)

5. Let $Y_1, \ldots, Y_n$ be i.i.d random variables such that for $0 < p < 1$, $P(Y_i = 1) = p$ and $P(Y_i = 0) = q = 1-p$. (Such random variables are called Bernoulli random variables.)

   a. Find the moment generating function for the Bernoulli random variable Y_1. Make sure you show your steps. (10pts)

   b. Find the moment generating function for $W = Y_1 + Y_2 + \ldots + Y_n$. Can you recognize which known distribution has this mgf? (10 pts)

Extra credit question. **(10 pts)**

6. If $Y_i, i = 1, 2$, are independent Gamma$(\alpha_i, 1)$ random variables, find the marginal distributions of $U_1 = \dfrac{Y_1}{Y_1 + Y_2}$ and $U_2 = \dfrac{Y_2}{Y_1 + Y_2}$.